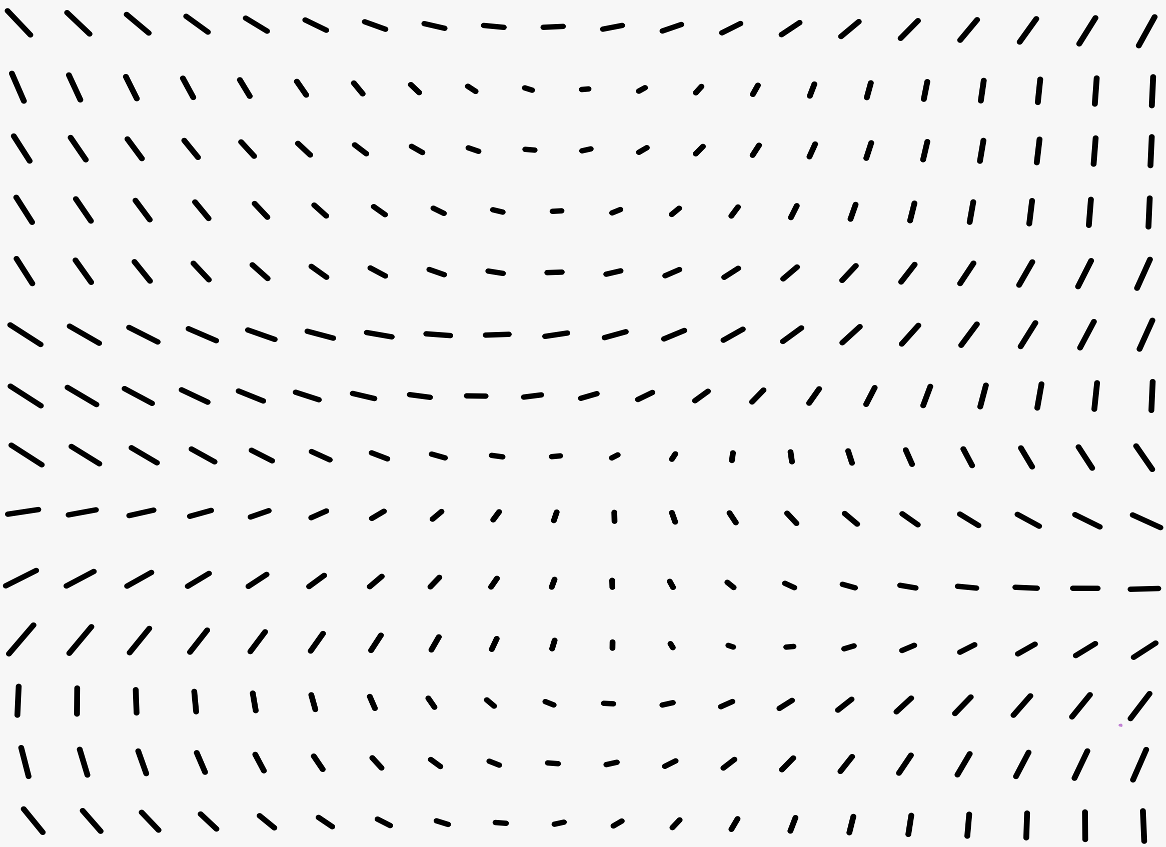
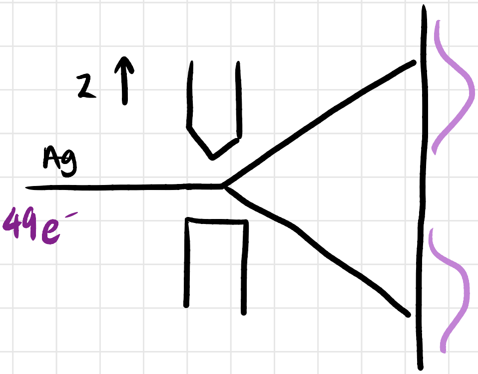
QUANTUM FIELD THEORY



Stern-Gerlach Experiment



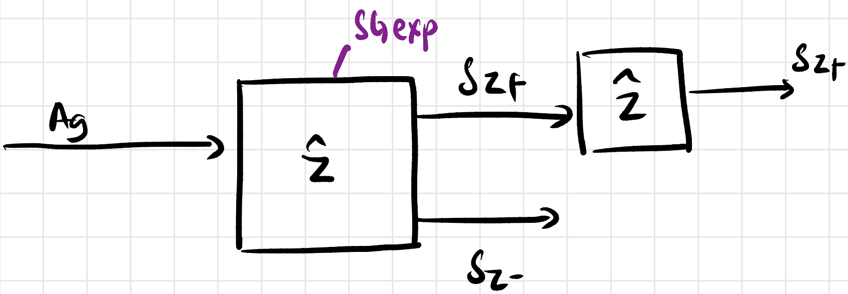
$$F_z = \mu_z \frac{\partial Q}{\partial z} \hat{z}$$

what letter?

$$\mu_z = -g\mu_B \vec{S_z}$$

spin angular momentum

$$S_{z+} = +\frac{\hbar}{2} \;;\; S_{z-} = -\frac{\hbar}{2}$$

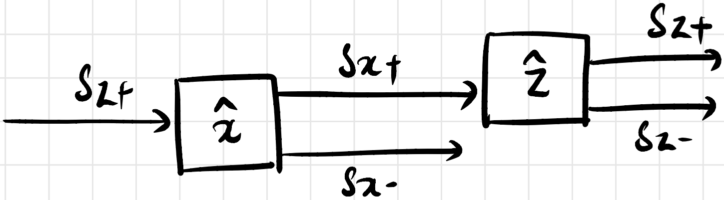


Bra-ket notation

- vectors
- $| \;\rangle \longrightarrow$ ket
- $\langle \;| \longrightarrow$ bra

$|S_{z+}\rangle$
$|S_{z-}\rangle$ states

- represents state in QM
- all info



- $|S_x+\rangle$ thought to be composed of only $|S_z+\rangle$
- However, we observe $|S_x+\rangle = \alpha|S_z+\rangle + \beta|S_z-\rangle$

$$|S_x+\rangle = \alpha|S_z+\rangle + \beta|S_z-\rangle$$

$$|S_x-\rangle = \gamma|S_z+\rangle + \delta|S_z-\rangle$$

- Any new direction it passes through causes it to split, almost 'forgetting' its previous states
- Superposition of two states

States in x-direction

for normalisation

$$|S_x+\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle + \frac{1}{\sqrt{2}}|S_z-\rangle$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle - \frac{1}{\sqrt{2}}|S_z-\rangle$$

These vectors are set in linear vector space

- Why + & - ?

Linear Vector Space

$$LVS = \{ |v_1\rangle, |v_2\rangle \ldots \}$$

$$\text{Field} = \{ \alpha_1, \alpha_2 \ldots \}$$ ← complex no.s

$$\alpha |v_1\rangle \longrightarrow$$ new vector

- info on $\hat{z}$ hidden in S_z
- basis set : smallest no. of vectors to span that space

$$|S_z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|S_z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

orthonormal basis

Orthonormal Basis

- Basis is a set of vectors using which any vectors can be created (linearly independent)
- let $|\psi\rangle$ and $|\phi\rangle$ be two vectors in a vector space
- for orthogonal, $\langle\phi|\psi\rangle = 0$ ← inner product (dot)
- for orthonormal,

$$\langle\phi|\phi\rangle = 1$$
$$\langle\psi|\psi\rangle = 1$$

unit length

- just like $\hat{i}$ and $\hat{j}$

- Basis = $\{v_1, v_2, v_3\}$

- Dimensionality: no. of ~~basis~~ linearly independent vectors

4-dimensions: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

- Complex LVS

For linear independence

let $|v_1\rangle, |v_2\rangle \ldots$ be basis vectors

$$|v\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \ldots$$

$$\alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \ldots = 0$$

only if

$$\alpha_1 = \alpha_2 = \ldots = 0$$

- why were 2 dimensions chosen?
- no. of degrees of freedom
- To represent direction in 3D space, only 2 variables required (θ and ϕ)

Ket and Bra

$$|S_{z+}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \text{ket}$$

$\downarrow$ transpose (complex)

$$\langle S_{z+}| = \begin{pmatrix} 1 & 0 \end{pmatrix} \longrightarrow \text{bra}$$

(complex conjugate)

$$|\phi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\langle \phi| = \begin{pmatrix} z_1^* & z_2^* \end{pmatrix}$$

Inner Product (Dot Product)

- (Hilbert space)

$$\langle S_{z+} | \cdot | S_{z+} \rangle = \langle S_{z+} | S_{z-} \rangle$$

always gives complex number

$\langle S_{z+} | S_{z-} \rangle$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \longrightarrow \text{unit vector}$$

$\langle S_{z+} | S_{z-} \rangle$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = 0 \longrightarrow \text{normal}$$

LVS

$$S = \{ |V_1\rangle, |V_2\rangle, |V_3\rangle \ldots \}$$

$$F = \{ \alpha_1, \alpha_2, \alpha_3 \ldots \}$$

$$V = \alpha_1 |V_1\rangle + \alpha_2 |V_2\rangle + \ldots \longrightarrow \text{closed under } S$$

$$\alpha_1 (\alpha_2 |V\rangle) = \alpha_1 \alpha_2 \vec{V}$$

Dual vector space

$$S = \{ \langle V_1|, \langle V_2|, \langle V_3|, \dots \}$$

for every $|V_1\rangle$, there exists $\langle V_1|$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad (\alpha_1^* \quad \alpha_2^*)$$

Distributive Law

$$\langle \phi | \alpha_1 \psi_1 + \alpha_2 \psi_2 \rangle$$

$$= \alpha_1 \langle \phi | \psi_1 \rangle + \alpha_2 \langle \phi | \psi_2 \rangle$$

Hilbert Space

- inner product should exist
- $\langle \phi | \psi \rangle \rightarrow$ complex no.
- $\langle \psi | \psi \rangle \geq 0 \leftarrow$ norm
- $\langle \phi | \psi \rangle = \langle \psi | \phi^* \rangle$
- $|\alpha \psi\rangle = \alpha |\psi\rangle$ and $\langle \psi \alpha | = \langle \psi | \alpha^*$

Back To SG exp.

$$|S_x+\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle + \frac{1}{\sqrt{2}}|S_z-\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle - \frac{1}{\sqrt{2}}|S_z-\rangle$$

$$= \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$|S_y+\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle + \frac{i}{\sqrt{2}}|S_z-\rangle = \begin{pmatrix}1/\sqrt{2}\\i/\sqrt{2}\end{pmatrix}$$

$$|S_y-\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle - \frac{i}{\sqrt{2}}|S_z-\rangle = \begin{pmatrix}1/\sqrt{2}\\-i/\sqrt{2}\end{pmatrix}$$

Eigenvalues

Matrix A $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\hat{A}\,|\psi\rangle = \lambda|\psi\rangle$$

set of $|\psi\rangle$ — eigen vectors

set of λ — eigen values

eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ find eigenvalues and eigenvectors

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 2^2 = 0$$

$$1-\lambda = \pm 2$$

$$\lambda = 1 \pm 2$$

$\lambda = -1, 3 \longrightarrow$ eigenvalues eigenvectors

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

case 1:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$2v_1 + 2v_2 = 0$$

$$v_1 = -v_2$$

eigenvectors: $\begin{pmatrix} x \\ -x \end{pmatrix}$ eg $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

case 2:

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -2v_1 + 2v_2 = 0$$

$$v_1 = v_2$$

eigenvectors: $\begin{pmatrix} x \\ x \end{pmatrix}$ eg $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- This means, any eigenvector × a vector = eigenvalue × the vector
- using this eg:

operator eigenvalue

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenvectors

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Every matrix has eigenvalues and eigenvectors
- Every observable in QM is a matrix (operator)

Operators of Spin States

eigenvalue

$$\hat{S}_z \, |S_{z+}\rangle = \pm\frac{\hbar}{2} \, |S_{z+}\rangle \longrightarrow (1)$$

let $\hat{S}_z = \frac{\hbar}{2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

eq (1)

4×4

$$\frac{\hbar}{2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \quad a_{11} = 1, \quad a_{21} = 0$$

$$\hat{S}_z \, |S_z-\rangle = \frac{-\hbar}{2} \, |S_z-\rangle$$

$$\frac{\hbar}{2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \quad a_{12} = 0, \quad a_{22} = 1$$

Pauli matrix $\hat{S}_z$ (operator)

first Pauli matrix

$$\boxed{\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

tend to leave out $\hbar/2$

To find $\hat{S}_x$

$$\hat{S}_x \, |S_x+\rangle = +1 \, |S_x+\rangle$$

$$|S_x+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{a_{11}}{\sqrt{2}} + \frac{a_{12}}{\sqrt{2}} \\ \frac{a_{21}}{\sqrt{2}} + \frac{a_{22}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$a_{11} + a_{12} = 1$$

$$a_{21} + a_{22} = 1$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = -\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$a_{11} - a_{12} = -1$$

$$a_{21} - a_{22} = 1$$

$a_{11} = 0$ $\quad$ $a_{21} = 1$

$a_{12} = 1$ $\quad$ $a_{22} = 0$

$$\hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To find $\hat{S}_y$

$$\hat{S}_y \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = +1 \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

$$\hat{S}_y \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = -1 \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

$a_{11} + i a_{12} = 1$ $\quad$ $a_{11} - i a_{12} = -1$

$a_{21} + i a_{22} = i$ $\quad$ $a_{21} - i a_{22} = i$

$a_{11} = 0$ $\quad$ $a_{21} = i$

$a_{12} = -i$ $\quad$ $a_{22} = 0$

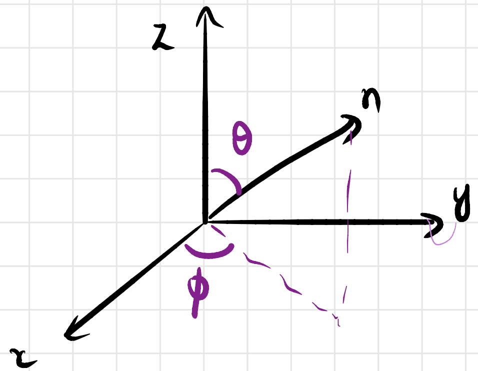
$$\boxed{\hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

$$\hat{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \hat{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrix

$$\{ \hat{S}_x, \hat{S}_y, \hat{S}_z, I \} = LVS$$

• Suppose $\vec{B}$ is along $\vec{n}$ in SG



how do you imagine?

$$\hat{S}_n = \vec{S} \cdot \vec{\sigma}$$

$$\vec{\sigma} = (n_x \; n_y \; n_z)$$

• $\hat{S}_x$ is a square matrix; how do you dot?

$$\hat{S}_n = \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z$$

$$\hat{S}_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} n_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} n_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} n_z$$

$n_x = n \sin\theta \cos\phi$; $n_y = n \sin\theta \sin\phi$; $n_z = n\cos\theta$

$$\hat{S}_n = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\hat{S}_n = \begin{pmatrix} n\cos\theta & n\sin\theta\cos\phi - i n \sin\theta \sin\phi \\ n\sin\theta\cos\phi + i n \sin\theta\sin\phi & -n\cos\theta \end{pmatrix}$$

$$\hat{S}_n = n \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & -\cos\theta \end{pmatrix}$$

magnitude not important

- use this operator on any vector in any direction to find eigenvalues and eigenvectors

- we can consider $n = 1$

$$\hat{S}_n = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

Hermitian Matrix

- Every observable is a Hermitian
- A matrix A is Hermitian iff

$$(A^*)^T = A$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \; ; \quad A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix}$$

$$(A^*)^T = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- diagnals always real, eigenvalues always real
- eg: spin matrix $\hat{S}_y$

$$\hat{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_y^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$(\hat{S}_y^*)^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hat{S}_y$$

read Leonard Susskind QM

$A = \langle S_{z-} | S_{z+} \rangle$ prob. amp. of find in S_{z-} given S_{z+}

$= 0$

$$A = \langle S_{z+} | S_{z+} \rangle = (1 \quad 0) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1/\sqrt{2}$$

$A^2 =$ prob. $= 1/2$

$$|A\rangle = \alpha_1 |A_1\rangle + \alpha_2 |A_2\rangle$$

prob $= \alpha_1^* \alpha + \alpha_2^* \alpha$

Gram-Schmidt Procedure

LVS : $(u_1, u_2, u_3 \ldots)$ — not orthonormal basis

Create orthonormal basis using $(u_1, u_2, u_3 \ldots)$

Orthonormal Basis $|a_n\rangle, |a_m\rangle \ldots$

$$\boxed{\begin{array}{ll} \langle a_n | a_m \rangle = 0 & , \; m \neq n \\ \langle a_n | a_m \rangle = 1 & , \; m = n \end{array}}$$

] kronecker delta function

δ_{mn}

Dirac Delta Function

$$\begin{array}{ll} \delta = 0 & , \; x \neq x_0 \\ \delta = \infty & , \; x = x_0 \end{array}$$

In Hibert Space (LVS), $\langle \phi | \psi \rangle > 0$

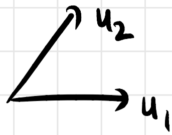
If ϕ and ψ are normal, $\langle \phi | \psi \rangle = 0$

The value of $\langle \phi | \psi \rangle$ cannot be negative

(u_1, u_2) — not orthonormal

$$\downarrow$$

$\{v_1, v_2\}$ — orthonormal



Take first basis vector $v_1 = u_1$

Find remaining orthonormal vector(s)

$$\vec{v_1} = \vec{u_1}$$

projection of u_2 on v_1

$$\vec{v_2} = \vec{u_2} - u_2 \cos\theta \, \hat{v_1}$$

$$= \vec{u_2} - \frac{\langle v_1 | u_2 \rangle}{v_1} \frac{|v_1\rangle}{v_1}$$

$$\vec{v_2} = \vec{u_2} - \frac{\langle v_1 | u_2 \rangle}{\langle v_1 | v_1 \rangle} |v_1\rangle$$

Q: Obtain orthonormal basis from $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$|v_1\rangle = |u_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|v_2\rangle = |u_2\rangle - \frac{\langle u_2 | v_1 \rangle}{\langle v_1 | v_1 \rangle} |v_1\rangle$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|v_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Normalisation

if $|\phi\rangle = \begin{pmatrix}a\\b\end{pmatrix}$

normalised $|\tilde{\phi}\rangle = \dfrac{|\phi\rangle}{\sqrt{\langle\phi|\phi\rangle}}$

$$|v\rangle = a_1|v_1\rangle + a_2|v_2\rangle + a_3|v_3\rangle$$

basis vectors

To obtain coefficients of bases,

$$\langle v_1|v\rangle = a_1$$

$$\langle v_2|v\rangle = a_2$$

$$\langle v_3|v\rangle = a_3$$

General vector $|v\rangle$

$$|v\rangle = \sum_i a_i |v_i\rangle$$

$$|v\rangle = \sum_i |v_i\rangle a_i$$

$$|v\rangle = \sum_i \boxed{|v_i\rangle\langle v_i|} |v\rangle$$

operated on $|v\rangle$ gives back $|v\rangle$

Projector Operator P

Hermitian $(A^*)^T$

$$\hat{P}^\dagger = \hat{P} \; ; \quad \underbrace{\hat{P}^2 = \hat{P}}_{\text{homework}}$$

$$\hat{G}^2 = (|v\rangle\langle v|)\cdot(|v\rangle\langle v|)$$
$$= |v\rangle\langle v| = \hat{G}^2$$

$$|\phi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\langle\psi|\phi\rangle = (\beta_1^* \;\; \beta_2^*)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underbrace{\alpha_1\beta_1^* + \alpha_2\beta_2^*}_{\text{a number}}$$

$$|\psi\rangle\langle\phi| = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{2\times1} (\alpha_1^* \;\; \alpha_2^*)_{1\times2} = \begin{pmatrix} \beta_1\alpha_1^* & \beta_1\alpha_2^* \\ \beta_2\alpha_1^* & \beta_2\alpha_2^* \end{pmatrix}$$

2x2 matrix (operator)

$|\psi\rangle\langle\phi|$ as a projection operator

$$\left(\begin{pmatrix} \beta_1\alpha_1^* & \beta_1\alpha_2^* \\ \beta_2\alpha_1^* & \beta_2\alpha_2^* \end{pmatrix}^*\right)^T$$

$A^\dagger = (A^*)^T = A \longrightarrow$ Hermitian (self-adjoint)

Eigenvalues of Hermitian Matrix are REAL and Eigen kets are orthogonal

eigenvalue

operator $\quad A|\psi_n\rangle = a_n|\psi_n\rangle$

eigen kets

$$A|\psi_m\rangle = a_m|\psi_m\rangle \longrightarrow (1)$$

$$\langle\psi_n|A^\dagger = a_n^* \langle\psi_n| \longrightarrow (2)$$

$(1) \times \langle\psi_n|$:

$$\langle\psi_n|A|\psi_m\rangle = a_m \langle\psi_n|\psi_m\rangle \longrightarrow (3)$$

$(2) \times |\psi_m\rangle$

$$\langle\psi_n|A^\dagger|\psi_m\rangle = a_n^* \langle\psi_n|\psi_m\rangle \longrightarrow (4)$$

subtracting (4) from (3)

$$(a_m - a_n^*) \langle \psi_n | \psi_m \rangle = 0$$

Case I

$m \neq n$

$$a_m = a_n^* \quad \text{or} \quad \langle \psi_n | \psi_m \rangle = 0$$

why?
homework

Case II

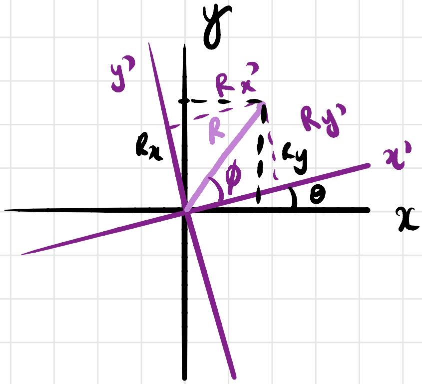
$m = n$

positive

$$(a_n - a_n^*) \langle \psi_n | \psi_n \rangle = 0$$

$a_n = a_n^* \longrightarrow$ a_n is real

Change of basis



$$\cos\phi = \frac{R_{x'}}{R}$$

$$\sin\phi = \frac{R_{y'}}{R}$$

$$\cos(\theta+\phi) = \frac{R_x}{R}$$

$$\sin(\theta+\phi) = \frac{R_y}{R}$$

$$R(\cos\theta\cos\phi - \sin\theta\sin\phi) = R_x$$

$$R(\sin\theta\cos\phi + \cos\theta\sin\phi) = R_y$$

Unitary Matrix

$$UU^{\dagger} = I$$

$$UU^{-1} = I$$

$$U^{\dagger} = U^{-1}$$

Commutator

$$[A, B] = AB - BA$$

Anti-Commutator

$$\{A, B\} = AB + BA$$

if $[A, B] = 0$, the operators A and B commute

if A & B are Hermitian and $AB - BA$ is Hermitian, then $[A, B] = 0$

$$(AB)^\dagger = B^\dagger A^\dagger$$

sum of diagonal elements = trace = 0

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\hat{S}_x \ \hat{S}_y] = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\frac{\hbar^2}{4}$$

$$= \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)\frac{\hbar^2}{4} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}\frac{\hbar^2}{4}$$

$$= \frac{\hbar}{2} \cdot 2i \ \hat{S}_z$$

$$[\hat{S}_x \ \hat{S}_y] = i\hbar \hat{S}_z$$

$$[\hat{S}_y \ \hat{S}_z] = \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \frac{\hbar}{2} \cdot 2i \ \hat{S}_x$$

$$[\hat{S}_y \ \hat{S}_z] = i\hbar \hat{S}_x$$

$$[\hat{S}_z \ \hat{S}_x] = \frac{\hbar^2}{4}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \frac{\hbar}{2} \cdot 2 \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \hbar \cdot \frac{\hbar}{2} i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[\hat{S}_z \ \hat{S}_x] = i\hbar \hat{S}_y$$

$$\hat{S} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\hat{S}^2 = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$[\hat{S}^2 \;\; \hat{S}_x] = \left(I\hat{S}_x - \hat{S}_x I \right) = 0$$

Quantum Computing

- factoring large numbers

Classical bit

states: 0 and 1

to represent a single state, one number needed

Qubit

states: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "on"

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "off"

computational basis vectors (S_z^+, S_z^-)

- both states are well-defined and measurable
- A qubit is represented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- To represent a single state, two numbers are needed.

Entanglement

tensor product

A $|0\rangle$
B $|0\rangle$

$$|0\rangle \otimes |0\rangle \quad \text{or } |00\rangle$$

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\ 0\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix}_{4\times1}$$

$$\begin{pmatrix}\alpha_1\\\beta_1\end{pmatrix} \otimes \begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}\\ \beta_1\begin{pmatrix}\alpha_2\\\beta_2\end{pmatrix}\end{pmatrix}$$

$$= \begin{bmatrix}1\\0\\0\\0\end{bmatrix}_{4\times1}$$

A $|0\rangle$
B $|1\rangle$

$$|01\rangle = \begin{bmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\ 0\begin{pmatrix}0\\1\end{pmatrix}\end{bmatrix}_{4\times1} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}_{4\times1}$$

A $|1\rangle$
B $|0\rangle$

$$|10\rangle = \begin{bmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\ 1\begin{pmatrix}1\\0\end{pmatrix}\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}_{4\times1}$$

A $|1\rangle$
B $|1\rangle$

$$|11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}_{4\times1}$$

Totally 4 entangled states.

2-qubit system

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

entangled states

- 4 complex no.s to represent a state
- Lives in 4D Hibert Space
- Classically, 2-bit systems need 2 no.s to specify state

3-qubit system

orthonormal

$$|\psi\rangle = \alpha|000\rangle + \beta|001\rangle + \gamma|010\rangle + \delta|011\rangle + \varepsilon|100\rangle + \chi|101\rangle + \phi|110\rangle + \omega|111\rangle$$

- 2^3 complex no.s to represent state
- 8-D vector space, 8 vectors (basis)

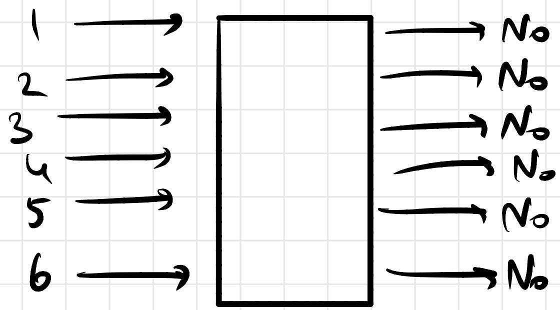
n-qubit system

- 2^n complex coefficients
- 2^n vectors
- 2^n dimensional vector space

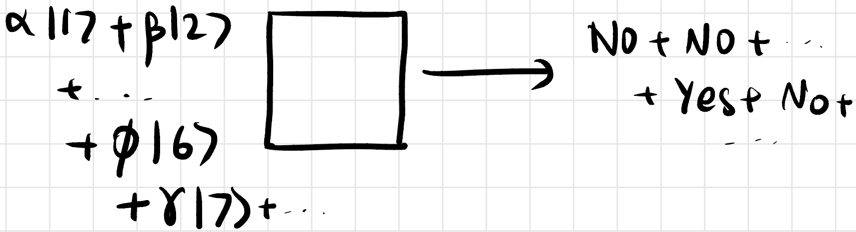
- to represent 10^{21} bits, 2^{70} qubits needed
- with 300 qubits, $2^{300} \times 10^{90} >$ no. of particles
- will collapse to n states

Logic Gates

- Check for a no. serially – classically
- check for 6



- Quantum



1) Single qubit gate

(i) Pauli $\hat{X}$ Gate

not gate

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \text{NOT}$$

$$\hat{X}\,|0\rangle \quad = \quad |1\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{x}\,|1\rangle = |0\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(ii) Pauli $\hat{z}$ Gate

flips $|1\rangle$ not $|0\rangle$

$$\hat{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{z}\,|0\rangle = |0\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{z}\,|1\rangle = -|1\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(iii) Hadamard Gate - most famous

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

Unitary matrix: $U^{\dagger} = U^{-1}$

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\hat{H}|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \;—\; |S_x^+\rangle$$

$$\hat{H}|1\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \leftarrow |S_x^-\rangle$$

- Every computation in QC needs a new algorithm

- In SG, if $\vec{B}$ applied on $\uparrow$ for certain t and ω, it can flip to $\downarrow$

Entanglement

separable

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

any measurement on $|\psi_1\rangle$ doesn't affect $|\psi_2\rangle$

$$|\psi_1\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|\psi_2\rangle = c|0\rangle - d|1\rangle = \begin{pmatrix} c \\ -d \end{pmatrix}$$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$= \begin{pmatrix} a \begin{pmatrix} c \\ -d \end{pmatrix} \\ b \begin{pmatrix} c \\ -d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ -ad \\ bc \\ -bd \end{pmatrix}$$

$$|\psi\rangle = ac|00\rangle - ad|01\rangle + bc|10\rangle - bd|11\rangle$$

- Suppose you measure ψ_1 and it is in $|0\rangle$
- $\therefore$ $a=1$, $b=0$
- $\therefore$ $|\psi\rangle = c|00\rangle - d|01\rangle$
  $= |0\rangle(c|0\rangle - d|1\rangle)$ unaffected
- $\therefore$ $|\psi_1\rangle$ and $|\psi_2\rangle$ are not entangled.

However,

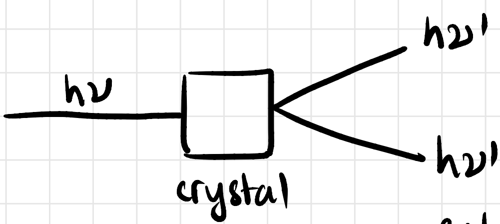
$$|\psi\rangle = a|\underset{A}{0}\underset{B}{0}\rangle + b|\underset{A}{1}\underset{B}{1}\rangle$$ entangled

$$P(A \rightarrow |0\rangle) = 1/2$$
$$P(A \rightarrow |1\rangle) = 1/2$$
] before measuring

Measure:

$$P(B \rightarrow |0\rangle) = 1$$
$$P(A \rightarrow |1\rangle) = 0$$

Spontaneous Parametric Down Conversion (SPDC)



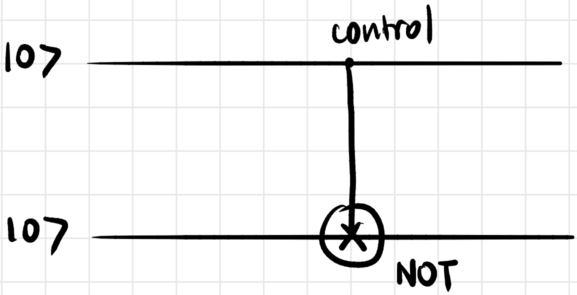
entanglement

Polarisation of 2 photons

Bell's Inequality

- Assume hidden variables
- Alan Guth

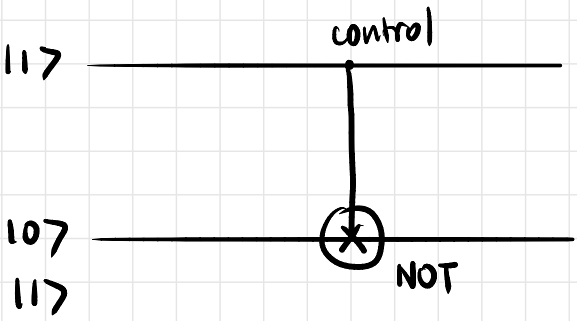
(iv) Controlled NOT gate



target

$|00\rangle$

control



$|11\rangle$

$|10\rangle$

$$|00\rangle \xrightarrow{CNOT} |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

CNOT + Hadamard $\longrightarrow$ universal

$$\text{CNOT}|00\rangle = \begin{pmatrix} 1 & \cdots \\ 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \end{pmatrix}_{4\times4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{4\times1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{4\times1}$$

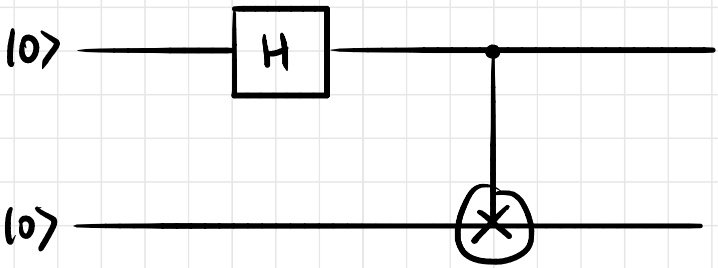
$$\text{CNOT}|01\rangle = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{CNOT}|10\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{CNOT}|11\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Write the truth table for this gate



$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

1-qubit system

$$\text{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)$$

2-qubit system

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Grover Algorithm

$$\{0 \ldots \ldots 2^n-1\}$$

$$f(x) = 0$$
$$f(x^*) = 1$$

$$\hat{O}|x\rangle = (-1)^{f(x)}|x\rangle$$
$$\hat{O}|x^*\rangle = (-1)^{f(x^*)}|x\rangle$$

$$\hat{D} = 2|s\rangle\langle s| - I$$

$$\hat{D}\hat{O}|\psi\rangle = |x^*\rangle$$

what we want ("yes")

go back to finding 6

superposition

check on IBM website